Chapter 2. Electrostatics

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***Introduction to Electrodynamics*, 3rd or 4rd Edition, David J. Griffiths**

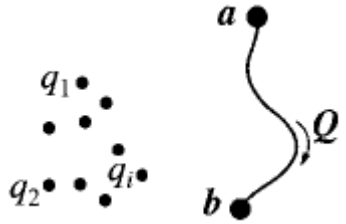
2.4 Work and Energy in Electrostatics

2.4.1 The Work Done to Move a Charge

To move a test charge ***Q*** from point **a** to point **b**, ***how much work*** will you have to do?

$$W = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l} = -Q \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} = Q[V(\mathbf{b}) - V(\mathbf{a})]$$

$\mathbf{F} = -Q\mathbf{E}$ (in opposite to electric force)



$$V(\mathbf{b}) - V(\mathbf{a}) = \frac{W}{Q}$$

➔ **The *potential difference between points* a *and* b is *equal to the work per unit charge required to carry a particle from* a *to* b.**

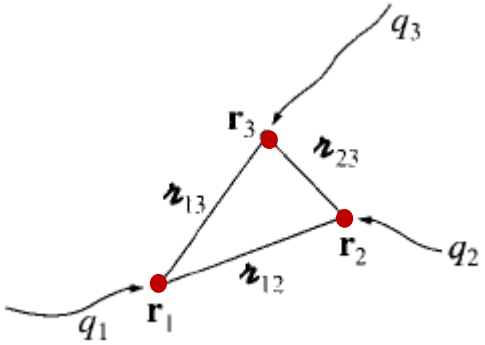
If you want to bring the charge Q in from far away and stick it at point **r,**

$$\xrightarrow{V(\mathbf{a}) = V(\infty) = 0} \boxed{W = QV(\mathbf{r})}$$

➔ ***Potential is potential energy per unit charge (just as the field is the force per unit charge).***

2.4.2 The Energy of a Point Charge Distribution

How much work would it take to assemble an entire collection of point charges?



The first charge, q_1, takes *no* work → $\boldsymbol{W_1 = 0}$

The second charge, q_2 → $W_2 = \dfrac{1}{4\pi\epsilon_0} q_2 \left(\dfrac{q_1}{r_{12}}\right)$

The third charge, q_3 → $W_3 = \dfrac{1}{4\pi\epsilon_0} q_3 \left(\dfrac{q_1}{r_{13}} + \dfrac{q_2}{r_{23}}\right) \longrightarrow W_4 = \dfrac{1}{4\pi\epsilon_0} q_4 \left(\dfrac{q_1}{r_{14}} + \dfrac{q_2}{r_{24}} + \dfrac{q_3}{r_{34}}\right)$

The *total* work necessary to assemble 4 charges,

$$W = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1q_2}{r_{12}} + \frac{q_1q_3}{r_{13}} + \frac{q_1q_4}{r_{14}} + \frac{q_2q_3}{r_{23}} + \frac{q_2q_4}{r_{24}} + \frac{q_3q_4}{r_{34}}\right)$$

$$\xrightarrow[\text{In general,}]{} W = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j>i}}^{n}\frac{q_iq_j}{r_{ij}} \xrightarrow{\text{Divided by 2}} W = \frac{1}{8\pi\epsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{q_iq_j}{r_{ij}}$$

$$\longrightarrow W = \frac{1}{2}\sum_{i=1}^{n} q_i \left(\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{4\pi\epsilon_0}\frac{q_j}{r_{ij}}\right) \longrightarrow \boxed{W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i)}$$

2.4.3 The Energy of a Continuous Charge Distribution

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i) \longrightarrow \boxed{W = \frac{1}{2}\int \rho V \, d\tau} \qquad \int \lambda V \, dl \qquad \int \sigma V \, da$$

There is a lovely way to rewrite this result in terms of E.

$$\rho = \epsilon_0 \nabla \cdot \mathbf{E}, \quad \text{so} \quad W = \frac{\epsilon_0}{2}\int (\nabla \cdot \mathbf{E}) V \, d\tau = \frac{\epsilon_0}{2}\left[-\int \mathbf{E} \cdot (\nabla V)\, d\tau + \oint V\mathbf{E} \cdot d\mathbf{a}\right] \quad \text{(Integration by parts)}$$

$$= \frac{\epsilon_0}{2}\left(\int_{\mathcal{V}} E^2 \, d\tau + \oint_{\mathcal{S}} V\mathbf{E} \cdot d\mathbf{a}\right) \longleftarrow \quad \nabla V = -\mathbf{E}$$

Note that the energy *W can defined, whatever* volume you use (as long as it encloses all the charge),

→ but the contribution from the ***volume integral of E^2*** **goes up**,

→ that of the ***surface integral of VE*** **goes down** since $E \sim 1/r^2$, $V \sim 1/r$, while da ~ r^2.

➔ ***For all space (r goes infinite), the surface integral goes to zero!***

Energy of Continuous Charge Distribution ⇒ $\boxed{W = \frac{1}{2}\int \rho V \, d\tau}$ ⇒ $\boxed{W = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2 \, d\tau}$

Example 2.8 Find the energy of a uniformly charged spherical shell of total charge *q* and radius *R*.

Solution 1: $W = \frac{1}{2}\int \sigma V \, da \xrightarrow{V = (1/4\pi\epsilon_0)q/R} W = \frac{1}{8\pi\epsilon_0}\frac{q}{R}\int \sigma \, da = \frac{1}{8\pi\epsilon_0}\frac{q^2}{R}$

Solution 2: Inside the sphere $\mathbf{E} = 0$; outside, $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$ $\quad W = \frac{\epsilon_0}{2}\int_{\text{outside}} E^2 \, d\tau = \frac{\epsilon_0}{2(4\pi\epsilon_0)^2}\int_{\text{outside}} \left(\frac{q^2}{r^4}\right)(r^2 \sin\theta \, dr \, d\theta \, d\phi) = \frac{1}{8\pi\epsilon_0}\frac{q^2}{R}$

2.4.4 Comments on Electrostatic Energy

(i) A perplexing "inconsistency."

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i) \qquad W = \frac{1}{2}\int \rho V \, d\tau$$

$$\longleftrightarrow \qquad W = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2 \, d\tau$$

Positive or negative depended on ρV

Always positive

➔ ***What's gone wrong?***
➔ ***Which equation is correct?***

$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i)$ ➔***It does not take into account*** the work necessary to *make* the point charges in the first place; we *started* with point charges and simply found the work required to bring them together.

$W = \frac{\epsilon_0}{2}\int_{\text{outside}} E^2 \, d\tau$ ➔For the first charge, it indicates that ***the energy of a single point charge is infinite*****.**

$$W = \frac{\epsilon_0}{2}\int_{\text{outside}} E^2 \, d\tau = \frac{\epsilon_0}{2(4\pi\epsilon_0)^2}\int_{\text{outside}} \left(\frac{q^2}{r^4}\right)(r^2 \sin\theta \, dr \, d\theta \, d\phi) = \frac{q^2}{8\pi\epsilon_0}\int_0^\infty \frac{1}{r^2}\, dr = \infty$$

➔**It is more *complete*,** in the sense that it tells you the *total* energy stored in a charge configuration (for example, imagine an electron.)

➔ For a continuous distribution there is no distinction, since the amount of charge *right at the point* is vanishingly small, and its contribution to the potential is zero.

➔ ***Therefore, all the equations are correct for a continuous distribution!***

Comments on Electrostatic Energy

(ii) Where is the energy stored?

$$W = \frac{1}{2}\int \rho V \, d\tau \quad \longleftrightarrow \quad W = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2 \, d\tau$$

Charge distribution
is confined in the volume

Electric field
is present everywhere *outside*

➔ ***Where is the energy, then?***
is it stored in the field?
Or is it stored in the charge?

It is unnecessary to worry about where the energy is located.

- → The difference is purely a matter of bookkeeping.
- → Total energy should be the same.
- → According to applications, we may choose either one of the equations.
  - → In radiation theory it is useful to regard the energy as being stored in the field, with a density

$$\frac{\epsilon_0}{2}E^2 = \text{energy per unit volume}$$

  - → But in electrostatics one could just say it is stored in the charge, with a density → $\frac{1}{2}\rho V$

(iii) The superposition principle.

Because electrostatic energy is ***quadratic*** in the fields, it does ***not obey a superposition principle***.
The energy of a compound system is *not* the sum of the energies of its parts considered separately.
There are also "cross terms":

$$W_{\text{tot}} = \frac{\epsilon_0}{2}\int E^2 \, d\tau = \frac{\epsilon_0}{2}\int (\mathbf{E}_1 + \mathbf{E}_2)^2 \, d\tau = \frac{\epsilon_0}{2}\int (E_1^2 + E_2^2 + 2\mathbf{E}_1 \cdot \mathbf{E}_2)\, d\tau = W_1 + W_2 + \epsilon_0 \int \mathbf{E}_1 \cdot \mathbf{E}_2 \, d\tau$$